

# Saturated simple and $k$ -simple topological graphs

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## Abstract

A *simple topological graph*  $G$  is a graph drawn in the plane so that any pair of edges have at most one point in common, which is either an endpoint or a proper crossing.  $G$  is called *saturated* if no further edge can be added without violating this condition. We construct saturated simple topological graphs with  $n$  vertices and  $O(n)$  edges. For every  $k > 1$ , we give similar constructions for  *$k$ -simple topological graphs*, that is, for graphs drawn in the plane so that any two edges have at most  $k$  points in common. We show that in any  $k$ -simple topological graph, any two independent vertices can be connected by a curve that crosses each of the original edges at most  $2k$  times. Another construction shows that the bound  $2k$  cannot be improved. Several other related problems are also considered.

## 1 Introduction

Saturation problems in graph theory have been studied at length, ever since the paper of Erdős, Hajnal, and Moon [2]. Given a graph  $H$ , a graph  $G$  is  $H$ -saturated if  $G$  does not contain  $H$  as a subgraph, but the addition of any edge joining two non-adjacent vertices of  $G$  creates a copy of  $H$ . The saturation number of  $H$ ,  $\text{sat}(n, H)$ , is the minimum number of edges in an  $H$ -saturated graph on  $n$  vertices. The saturation number for complete graphs was determined in [2]. A systematic study by Kászonyi and Tuza [8] found the best known general upper bound for  $\text{sat}(n, H)$  in terms of the independence number of  $H$ . The saturation number is now known, often precisely, for many graphs; for these results and related problems in graph theory we refer the reader to the thorough survey of J. Faudree, R. Faudree, and Schmitt [3].

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It is worth noting that  $\text{sat}(n, H) = O(n)$ , quite unlike the Turán function  $\text{ex}(n, H)$ , which is often superlinear.

In this paper, we study a saturation problem for *drawings* of graphs. In a drawing of a simple undirected graph  $G$  in the plane, every vertex is represented by a point, and every edge is represented by a curve between the points that correspond to its endpoints. If it does not lead to confusion, these points and curves are also called *vertices* and *edges*. We assume that in a drawing no edge passes through a vertex and no two edges are tangent to each other. A graph, together with its drawing, is called a *simple topological graph* if any two edges have at most one point in common, which is either their common endpoint or a proper crossing. In general, for any positive integer  $k$ , it is called a  *$k$ -simple topological graph* if any two edges have at most  $k$  points in common. We also assume that in a  $k$ -simple topological graph no edge crosses itself. Obviously, a 1-simple topological graph is a simple topological graph.

Our motivation partly comes from the following problem: At least how many pairwise disjoint edges can one find in every simple topological graph with  $n$  vertices and  $m$  edges [10]? (Note that the simplicity condition is essential here, as there are complete topological graphs on  $n$  vertices and no *two* disjoint edges, in which every pair of edges intersect at most twice [11].) For *complete* simple topological graphs, i.e., when  $m = \binom{n}{2}$ , Pach and Tóth conjectured ([1], page 398) that one can always find  $\Omega(n^\delta)$  disjoint edges for a suitable constant  $\delta > 0$ . This was shown by Suk [14] with  $\delta = 1/3$ ; see [4] for an alternative proof. Recently, Ruiz-Vargas [12] has improved this bound to  $\Omega(\sqrt{n/\log n})$ . Unfortunately, all known proofs break down for *non-complete* simple topological graphs. For *dense* graphs, i.e., when  $m \geq \varepsilon n^2$  for some  $\varepsilon > 0$ , Fox and Sudakov [5] established the existence of  $\Omega(\log^{1+\gamma} n)$  pairwise disjoint edges, with  $\gamma \approx 1/50$ . However, if  $m \ll n^2$ , the best known lower bound, due to Pach and Tóth [11], is only  $\Omega((\log m - \log n)/\log \log n)$ .

We know a great deal about the structure of complete simple topological graphs, but in the non-complete case our knowledge is rather lacunary. We may try to extend a simple topological graph to a complete one by adding extra edges and then explore the structural information we have for complete graphs. The results in the present note suggest that this approach is not likely to succeed: there exist very sparse simple topological graphs to which no edge can be added without violating the simplicity condition.

A  $k$ -simple, non-complete topological graph is *saturated* if no further edge can be added so that the resulting drawing is still a  $k$ -simple topological graph. In other words, if we connect any two non-adjacent vertices by a curve, it will have at least  $k + 1$  common points with one of the existing edges.

Consider the simple topological graph  $G_1$  with eight vertices, depicted in Figure 1. It is easy to verify that the vertices  $x$  and  $y$  cannot be joined by a new edge so that the resulting topological graph remains simple. Indeed, every edge of  $G_1$  is incident either to  $x$  or to  $y$ , and any curve joining  $x$  and  $y$  must cross at least one edge. On the other hand,  $G_1$  can be extended to a (saturated) simple topological graph in which every pair of vertices except  $x$  and  $y$  are connected by an edge.

Another example was found independently by Kynčl [7, Fig. 9]: The simple topological graph  $G_2$  depicted in Figure 2 has only six vertices, from which  $x$  and  $y$  cannot be joined by an edge without intersecting one of the original edges at least twice. Again,  $G_2$  can be extended to a simple topological graph in which every pair of vertices except  $x$  and  $y$  are connected by an edge.

In view of the fact that the graphs shown in Figures 1 and 2 can be extended to nearly

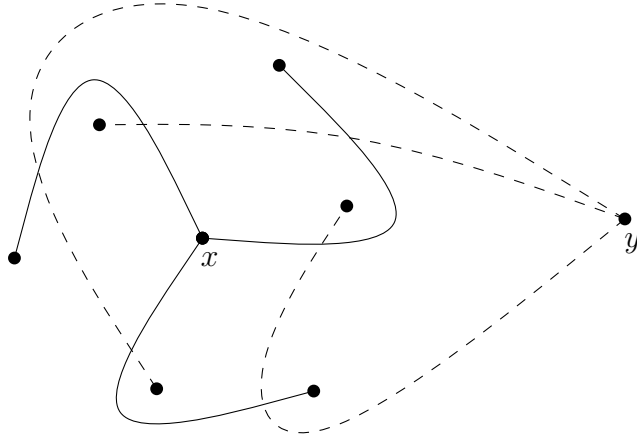


Figure 1: A topological graph  $G_1$ : the edge  $\{x, y\}$  cannot be added.

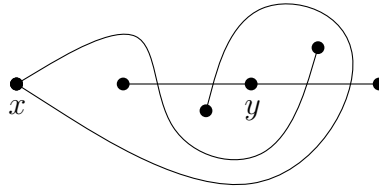


Figure 2: A topological graph  $G_2$ : the edge  $\{x, y\}$  cannot be added.

complete simple topological graphs, it is a natural question to ask whether every saturated simple topological graph with  $n$  vertices must have  $\Omega(n^2)$  edges. It is not obvious at all, whether there exist saturated non-complete  $k$ -simple topological graphs for some  $k > 1$ . Our next theorem shows that there are such graphs, for every  $k$ , moreover, they may have only a linear number of edges.

**Theorem 1.** *For any positive integers  $k$  and  $n \geq 4$ , let  $s_k(n)$  be the minimum number of edges that a saturated  $k$ -simple topological graph on  $n$  vertices can have. Then*

(i) *we have*

$$1.5n \leq s_1(n) \leq 17.5n,$$

(ii) *for  $k > 1$  we have*

$$n \leq s_k(n) \leq 16n.$$

For our best upper bounds see Table 1.

$k$	1	2	3	4	5	6	7	8	9	10	$\geq 11$
upper bound	$17.5n$	$16n$	$14.5n$	$13.5n$	$13n$	$9.5n$	$10n$	$9.5n$	$7n$	$9.5n$	$7n$

Table 1: Upper bounds on the minimum number of edges in saturated  $k$ -simple topological graphs.

For any positive integers  $k$  and  $l$ ,  $k < l$ , a topological graph  $G$  together with a pair of non-adjacent vertices  $\{u, v\}$  is called a  $(k, l)$ -construction if  $G$  is  $k$ -simple and any curve joining  $u$  and  $v$  has at least  $l$  points in common with at least one edge of  $G$ . Using this terminology, every saturated non-complete  $k$ -simple topological graph together with any pair of non-adjacent vertices is a  $(k, k + 1)$ -construction.

**Theorem 2.** *For every  $k > 0$ ,*

- (i) *There exists a  $(k, 2k)$ -construction,*
- (ii) *There is no  $(k, l)$ -construction with  $l > 2k$ .*

For any positive integers  $k$  and  $l$ ,  $k < l$ , a *non-complete* topological graph  $G$  is called  $(k, l)$ -saturated if  $G$  is  $k$ -simple and any curve joining *any pair* of non-adjacent vertices has at least  $l$  points in common with at least one edge of  $G$ . Obviously, every saturated  $k$ -simple topological graph is  $(k, k + 1)$ -saturated. Clearly, every  $(k, l)$ -saturated topological graph, together with any pair of its non-adjacent vertices, is a  $(k, l)$ -construction. However, for  $l > k + 1$ , the existence of a  $(k, l)$ -construction does not necessarily imply the existence of a  $(k, l)$ -saturated topological graph. The best we could prove is the following.

**Theorem 3.** *For any  $k > 0$ , there exists a  $(k, \lceil 3k/2 \rceil)$ -saturated topological graph.*

In the proof of Theorem 2 we obtain a set of six curves, any two of which cross at most once, and two points, such that any curve connecting them has to cross one of the six curves at least twice (see Figure 10).

On the contrary, it follows from Levi's enlargement lemma [9] that if the curves have to be *bi-infinite*, that is, two-way unbounded, then there is no such construction. A *pseudoline arrangement* is a set of bi-infinite curves such that any two of them cross exactly once. By Levi's lemma, for any two points not on the same line, the arrangement can be extended by a pseudoline through these two points. A *k-pseudoline arrangement* is a set of bi-infinite curves such that any two of them cross at most  $k$  times. A *k-pseudocircle arrangement* is a set of closed curves such that any two of them cross at most  $k$  times. Elements of pseudoline arrangements and  $k$ -pseudoline arrangements are called *pseudolines*. Note that for even  $k$ ,  $k$ -pseudoline arrangements can be considered a special case of  $k$ -pseudocircle arrangements.

Snoeyink and Hershberger [13] generalized Levi's lemma to 2-pseudoline arrangements and 2-pseudocircle arrangements as follows. They proved that for every 2-pseudocircle arrangement and three points, not all on the same pseudocircle, the arrangement can be extended by a closed curve through these three points so that it remains a 2-pseudocircle arrangement. They also showed that for  $k \geq 3$ , an analogous statement with  $k$ -pseudoline arrangements and  $k + 1$  given points is false.

A  $k$ -pseudoline arrangement is  $(p, l)$ -forcing if there is a set  $A$  of  $p$  points such that every bi-infinite curve through the points of  $A$  crosses one of the pseudolines at least  $l$  times. Snoeyink and Hershberger [13] found  $(k + 1, k + 1)$ -forcing  $k$ -pseudoline arrangements for  $k \geq 3$ . We generalize their result as follows.

**Theorem 4.** (i) *For every  $k \geq 1$ , there is a  $(3, 5\lceil (k - 7)/4 \rceil)$ -forcing  $k$ -pseudoline arrangement.*

- (ii) *For every  $k \geq 1$ , there is a  $(k, \Omega(k \log k))$ -forcing  $k$ -pseudoline arrangement.*

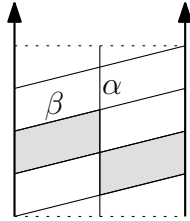


Figure 3: A 4-spiral formed by  $\alpha$  and  $\beta$ . One of the four regions is shaded. The arrows represent the two edges of the rectangle that are glued together to form a cylinder.

In Section 2 we define tools necessary for our constructions. In Section 2.1 we define *spirals* and use them in Lemma 6 to prove the existence of a  $(k, \lceil 7(k-1)/6 \rceil)$ -construction (for  $k \geq 8$ ). Although Lemma 6 is a very weak version of Theorem 2, its proof is a relatively simple construction, which serves as the basis of all our further constructions. In Section 2.2 we define another tool, *forcing blocks*, and as an illustration, we prove Lemma 9, which is an improvement of Lemma 6, yet still weaker than Theorem 2. Finally, in Section 2.3, we define the remaining necessary tools, *grid blocks* and *double-k-forcing blocks*, and use them to prove Theorem 2 (i).

In Section 3 we prove Theorem 2 (ii). Our proof is self-contained and independent of the tools developed in Section 2. In Section 4 we prove the upper bounds in Theorem 1, by giving five different constructions; the first one is for  $k = 1$ , and it is essentially different from the other four, which use spirals, grid blocks, and forcing blocks described previously in Section 2. In Section 5 we prove the lower bounds in Theorem 1. Our proof is self-contained and independent of the remaining sections. In Section 6, we prove Theorem 3. Our construction uses grid blocks and double-forcing blocks described in Section 2. In Section 7, we prove Theorem 4. Our constructions use spirals from Section 2. We finish the paper with some remarks and open problems.

## 2 Building blocks for $(k, l)$ -constructions

With the exception of the proof of Theorem 1 (i), we construct drawings on a vertical cylinder, which can be transformed into a planar drawing. The cylinder will be represented by an axis-parallel rectangle whose left and right vertical sides are identified. Curves on the cylinder are also represented in the axis-parallel rectangle, where they can “jump” between the left and the right sides. Edges will be drawn as  $y$ -monotone curves.

Drawings will be constructed from *blocks*. Each block is a horizontal “slice” of the cylinder, represented again by an axis-parallel rectangle, say,  $R$ , whose left and right vertical sides are identified. A *cable* in a block is a group of intervals of edges that go very close to each other but do not cross in the block. A cable is represented by a single curve which goes very close to each edge in the cable. A curve or a cable in a block  $B$  whose endpoints are on the top and bottom boundary of  $R$  is called a *transversal* of  $B$ . For any curves or cables  $a$  and  $b$ , let  $\text{CR}(a, b)$  denote the number of crossings between  $a$  and  $b$ .

## 2.1 Spirals and a $(k, \lceil 7(k-1)/6 \rceil)$ -construction

Let  $B$  be a block on the cylinder, represented by the unit square  $R$  with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$ ,  $(0, 1)$ , the two vertical sides are identified. Let  $\alpha$  be a straight line segment from  $(0.5, 0)$  to  $(0.5, 1)$ , and let  $\beta$  be represented as the union of  $m$  straight line segments,  $b_1, b_2, \dots, b_m$ , where  $b_i$  is the segment from  $(0, (i-1)/m)$  to  $(1, i/m)$ . See Figure 3. Cables  $a$  and  $b$  in  $B$  form an  $m$ -spiral if there is a homeomorphism of  $B$  that takes  $a$  to  $\alpha$  and  $b$  to  $\beta$  and maps the lower boundary of  $B$  to itself. Clearly, such cables  $a$  and  $b$  are transversals of  $B$  and intersect exactly  $m$  times.

**Observation 5.** *Suppose that  $a$  and  $b$  form an  $m$ -spiral in a block  $B$ . Then every transversal of  $B$  crosses  $a$  and  $b$  together at least  $m - 1$  times.*

*Proof.* Let  $\kappa$  be a transversal of  $B$ . Extend  $B$  to a two-way infinite cylinder,  $B'$ . Cables  $a$  and  $b$  together divide the cylinder  $B'$  into  $m$  regions, say,  $B_1, B_2, \dots, B_m$ , from bottom to top. One endpoint of  $\kappa$  is in  $B_1$ , the other one is in  $B_m$ . It is easy to see that  $B_i$  and  $B_j$  have a common boundary if and only if  $|i - j| = 1$ . Therefore, to go from  $B_1$  to  $B_m$ ,  $\kappa$  has to cross at least  $m - 1$  boundaries.  $\square$

Using spirals, we are able to prove the following weak version of Theorem 2 (i).

**Lemma 6.** *For  $k \geq 8$  there exists a  $(k, l)$ -construction with  $l = \lceil 7(k-1)/6 \rceil > k$ .*

*Proof.* The construction consists of 7 consecutive blocks,  $X, A, B, C, D, E, Y$ , in this order (say, from bottom to top). First we define six independent edges,  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ , and two isolated vertices,  $x$  and  $y$ .

Put  $x$  in  $X$  and  $y$  in  $Y$ . The edges  $\alpha_1$  and  $\alpha_2$  are in the blocks  $A$  and  $B$ , both have one endpoint on the boundary of  $X$  and  $A$  and one on the boundary of  $B$  and  $C$ . Edges  $\beta_1$  and  $\beta_2$  are in  $B, C$  and  $D$ , both have one endpoint on the boundary of  $A$  and  $B$  and one on the boundary of  $D$  and  $E$ . Edges  $\gamma_1$  and  $\gamma_2$  are in  $D$  and  $E$ , both have one endpoint on the boundary of  $C$  and  $D$  and one on the boundary of  $E$  and  $Y$ . The edges  $\alpha_1$  and  $\alpha_2$  form a  $k$ -spiral in  $A$  and a cable in  $B$ . The edges  $\beta_1$  and  $\beta_2$  form another cable in  $B$ , and these two cables form a  $k$ -spiral. Further,  $\beta_1$  and  $\beta_2$  form a  $k$ -spiral in  $C$  and a cable in  $D$ . The edges  $\gamma_1$  and  $\gamma_2$  form another cable in  $D$ , and these two cables form a  $k$ -spiral. Finally,  $\gamma_1$  and  $\gamma_2$  form a  $k$ -spiral in  $E$ .

We show that every curve  $\kappa$  from  $x$  to  $y$  crosses one of the curves  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$  at least  $7(k-1)/6$  times. Let  $\kappa$  be a fixed curve from  $x$  to  $y$ . For every  $\chi \in \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\}$  and  $Z \in \{A, B, C, D, E\}$ , let  $Z(\chi)$  denote the number of intersections of  $\chi$  with  $\kappa$  in  $Z$ . (That is,  $A(\alpha_1)$  is the number of intersections between  $\alpha_1$  and  $\kappa$  in  $A$ .) By Observation 5, we have

$$\begin{aligned} A(\alpha_1) + A(\alpha_2) &\geq k - 1, \\ B(\alpha_i) + B(\beta_j) &\geq k - 1, \quad 1 \leq i, j \leq 2, \\ C(\beta_1) + C(\beta_2) &\geq k - 1, \\ D(\beta_i) + D(\gamma_j) &\geq k - 1, \quad 1 \leq i, j \leq 2, \\ E(\gamma_1) + E(\gamma_2) &\geq k - 1. \end{aligned}$$

Therefore, we can assume without loss of generality that  $A(\alpha_1) \geq (k-1)/2$ ,  $C(\beta_1) \geq (k-1)/2$  and  $E(\gamma_1) \geq (k-1)/2$ .

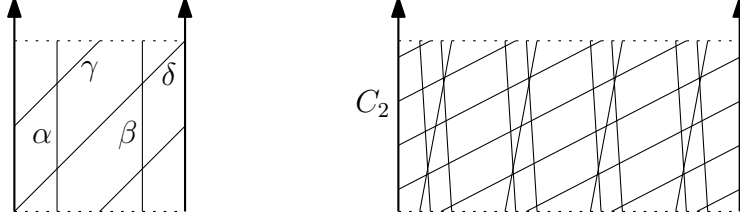


Figure 4: Left: a crossing-forcing configuration formed by curves  $\alpha, \beta, \gamma, \delta$ . Right: cables in the subblock  $C_2$ .

It follows that

$$\begin{aligned} & \text{CR}(\alpha_1, \kappa) + \text{CR}(\beta_1, \kappa) + \text{CR}(\gamma_1, \kappa) \\ &= A(\alpha_1) + B(\alpha_1) + B(\beta_1) + C(\beta_1) + D(\beta_1) + D(\gamma_1) + E(\gamma_1) \geq 7(k-1)/2. \end{aligned}$$

Consequently, at least one of  $\text{CR}(\alpha_1, \kappa)$ ,  $\text{CR}(\beta_1, \kappa)$ ,  $\text{CR}(\gamma_1, \kappa)$  is at least  $7(k-1)/6 > k$ , since  $k \geq 8$ .  $\square$

## 2.2 Forcing blocks and a $(k, 2k-1)$ -construction

We define another type of block, which is built from several subblocks. Let  $B$  be a block on the cylinder represented by the unit square  $R$  with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ ,  $(1, 1)$ , where the two vertical sides are identified. See Figure 4, left. Let  $\alpha$  be a straight line segment from  $(0.25, 0)$  to  $(0.25, 1)$ ,  $\beta$  a straight line segment from  $(0.75, 0)$  to  $(0.75, 1)$ ,  $\delta$  a straight line segment from  $(0, 0)$  to  $(1, 1)$ , and let  $\gamma$  be represented as the union of the segment from  $(0.5, 0)$  to  $(1, 0.5)$  and the segment from  $(0, 0.5)$  to  $(0.5, 1)$ . Cables  $a, b, c$  and  $d$  in a block  $B$  form a *crossing-forcing configuration* if there is a homeomorphism of  $B$  that takes  $a$  to  $\alpha$ ,  $b$  to  $\beta$ ,  $c$  to  $\gamma$ , and  $d$  to  $\delta$ , and maps the lower boundary of  $B$  to itself. Clearly, in this case any two cables intersect at most once.

**Observation 7.** *Suppose that cables  $a, b, c$  and  $d$  form a crossing-forcing configuration in a block  $B$ . Then every transversal of  $B$  crosses at least one of  $a, b, c$ , or  $d$ .*  $\square$

Fix  $k > 0$ . Now we define the  $k$ -forcing block  $B_k$  of  $4^k$  edges,  $a_1, a_2, \dots, a_m$ ,  $m = 4^k$ . We build  $B_k$  from  $k$  subblocks  $C_1, C_2, \dots, C_k$ , arranged from top to bottom in this order. In  $C_1$ , divide our edges into four equal subsets, each form a cable, and these four cables form a crossing-forcing configuration in  $C_1$ . In general, suppose that for some  $i$ ,  $1 \leq i < k$ ,  $C_i$  contains  $4^i$  cables  $c_1, c_2, \dots, c_{4^i}$  and each of them contains  $4^{k-i}$  edges. For each cable  $c_j$  of  $C_i$ , divide the corresponding set of edges into four equal subsets, each of them form a cable in  $C_{i+1}$ , and let these four cables form a crossing-forcing configuration in  $C_{i+1}$ . It is possible to draw the cables so that any two of them intersect at most once in  $C_{i+1}$  and so that for every two edges  $e \in c_j$  and  $f \in c_{j'}$ ,  $j < j'$ , the edges  $e$  and  $f$  intersect the top and the bottom boundary of  $C_{i+1}$  in the same order. See Figure 4, right. Clearly,  $C_{i+1}$  contains  $4^{i+1}$  cables and each of them contains  $4^{k-i-1}$  edges.

The resulting block  $B_k = \cup_{i=1}^k C_i$  is called a  $k$ -forcing block of edges  $a_1, a_2, \dots, a_m$ , where  $m = 4^k$ . The next lemma explains the name.



**Lemma 8.** *Suppose that  $B_k = \cup_{i=1}^k C_i$  is a  $k$ -forcing block of edges  $a_1, a_2, \dots, a_m, m = 4^k$ . Then every transversal of  $B_k$  intersects at least one of  $a_1, a_2, \dots, a_m$  at least  $k$  times.*

*Proof.* We prove the statement by induction on  $k$ . For  $k = 1$  the statement is equivalent to Observation 7. Suppose that the statement has been proved for  $k - 1$ , and let  $\kappa$  be a transversal of  $B_k$ . By Observation 7,  $\kappa$  crosses at least one of the cables in  $C_1$ . Consider now only the  $4^{k-1}$  edges that belong to that cable. These edges form a  $(k - 1)$ -forcing block in  $B'_{k-1} = \cup_{i=2}^k C_i$ . Therefore, by the induction hypothesis,  $\kappa$  crosses one of the edges at least  $k - 1$  times in  $B'_{k-1}$ . It also crosses this edge in  $C_1$ , so we are done.  $\square$

Now we prove a statement which is slightly weaker than Theorem 2 (i), but much stronger than Lemma 6.

**Lemma 9.** *For every  $k > 0$ , there exists a  $(k, 2k - 1)$ -construction.*

*Proof.* The construction consists of  $2k + 1$  consecutive blocks,  $X, F_1, S_1, F_2, S_2, \dots, S_{k-1}, F_k, Y$ , in this order (from bottom to top). Let  $m = 4^k$ . We define  $km$  independent edges  $\alpha_i^j$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq m$ , and two isolated vertices  $x$  and  $y$  as follows. Put  $x$  in  $X$  and  $y$  in  $Y$ .

- The edges  $\alpha_1^j$ ,  $1 \leq j \leq m$ , are in the blocks  $F_1$  and  $S_1$ ,
- for every  $i$ ,  $1 < i < k$ , the edges  $\alpha_i^j$ ,  $1 \leq j \leq m$ , are in the blocks  $S_{i-1}$ ,  $F_i$  and  $S_i$ ,
- the edges  $\alpha_k^j$ ,  $1 \leq j \leq m$ , are in the blocks  $S_{k-1}$  and  $F_k$ .

For every  $i$ ,  $1 \leq i \leq k$ , the edges  $\alpha_i^j$ ,  $1 \leq j \leq m$ , form a  $k$ -forcing block in  $F_i$ . For every  $i$ ,  $1 \leq i \leq k - 1$ , the edges  $\alpha_i^j$ ,  $1 \leq j \leq m$ , form one cable in  $S_i$ , the edges  $\alpha_{i+1}^j$ ,  $1 \leq j \leq m$ , form another cable in  $S_i$ , and these two cables form a  $k$ -spiral in  $S_i$ .

Let  $\kappa$  be a fixed curve from  $x$  to  $y$ . We show that  $\kappa$  crosses one of the curves  $\alpha_i^j$  at least  $2k - 1$  times. For every  $i$ ,  $1 \leq i \leq k$ , by Lemma 8, there is a  $j$ ,  $1 \leq j \leq m$ , such that  $\alpha_i^j$  and  $\kappa$  cross at least  $k$  times in  $F_i$ . Denote this  $\alpha_i^j$  by  $\alpha_i$ .

For every  $Z \in \{F_1, F_2, \dots, F_k, S_1, S_2, \dots, S_{k-1}\}$  and  $1 \leq i \leq k$ , let  $Z(\alpha_i)$  denote the number of intersections of  $\alpha_i$  with  $\kappa$  in the block  $Z$ . By the choice of  $\alpha_i$ , for every  $i$ ,  $1 \leq i \leq k$ , we have

$$F_i(\alpha_i) \geq k.$$

By Observation 5, for every  $i$ ,  $1 \leq i \leq k - 1$ , we have

$$S_i(\alpha_i) + S_i(\alpha_{i+1}) \geq k - 1.$$

Summing up,

$$\begin{aligned} \sum_{i=1}^k \text{CR}(\kappa, \alpha_i) &= \sum_{i=1}^k F_i(\alpha_i) + \sum_{i=1}^{k-1} (S_i(\alpha_i) + S_i(\alpha_{i+1})) \\ &\geq k^2 + (k - 1)^2 = (2k - 2)k + 1. \end{aligned}$$

Therefore, for some  $i$ ,  $\text{CR}(\kappa, \alpha_i) \geq 2k - 1$ .  $\square$



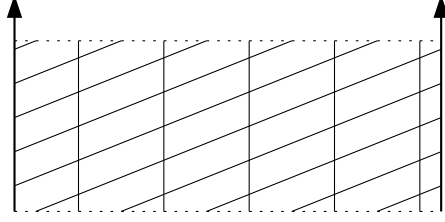


Figure 5: A  $(5, 1)$ -grid block  $G(5, 1)$ .

## 2.3 Grid blocks, Double-forcing blocks, and a proof of Theorem 2 (i)

### Grid blocks.

Let  $m, k > 0$ . An  $(m, 1)$ -grid block,  $G(m, 1)$  contains two groups,  $G'$  and  $G''$ , of cables (or edges). Both  $G'$  and  $G''$  contain  $m$  cables. Refer to Figure 5. The cables of  $G'$  form  $m$  parallel segments in  $G(m, 1)$ . The cables of  $G''$  are also parallel in  $G(m, 1)$  but make exactly one twist around the cylinder, intersecting every cable of  $G'$  exactly once. Moreover, the cables from  $G'$  and  $G''$  intersect both the upper and lower boundary alternately. An  $(m, k)$ -grid block  $G(m, k)$  consists of  $k$  identical subblocks  $G(m, 1)$  stacked on top of each other.

Observe that  $G(2, 1)$  is a crossing-forcing configuration and that  $G(1, k)$  is a  $k$ -spiral. So grid blocks are common generalizations of the spirals and crossing-forcing blocks.

The following observation generalizes Observation 5 and is easily shown by induction.

**Observation 10.** *Every transversal of the grid block  $G(m, k)$  has at least  $mk - 1$  crossings with the cables in  $G(m, k)$ .*  $\square$

### Double- $k$ -forcing blocks.

Let  $k > 0$ . A double- $k$ -forcing block  $D_k$  contains two groups of cables (edges), say,  $D'$  and  $D''$ .

- Each of  $D'$  and  $D''$  contains  $4^k$  cables forming a  $k$ -forcing block in  $B$ .
- The cables of  $D'$  and  $D''$  are consecutive along the upper and lower boundaries (but they are ordered differently on the two boundaries).
- Any two cables in  $D_k$  intersect at most  $k$  times.

We can construct double- $k$ -forcing blocks from subblocks in the same way as  $k$ -forcing blocks. For  $k = 1$ , the construction is shown on Figure 6. Suppose we already have  $D_i$ . We add a subblock  $C'_{i+1}$  to the bottom of  $D_i$  as follows. We divide each cable into four subcables and let these cables form a crossing-forcing configuration in  $C'_{i+1}$ , so that any two cables cross at most once in  $C'_{i+1}$  and the subcables of the same cable are consecutive along the upper and lower boundary of  $C'_{i+1}$ .

The following is a direct consequence of Lemma 8.

**Observation 11.** *Suppose that  $D_k$  is a double- $k$ -forcing block with two groups of cables  $D'$  and  $D''$ . Then for any transversal  $\kappa$  of  $D_k$ , there are cables  $\alpha' \in D'$  and  $\alpha'' \in D''$  that both cross  $\kappa$  at least  $k$  times.*

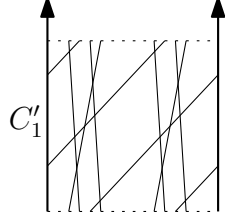


Figure 6: A double-1-forcing block.

Now, we are ready to prove Theorem 2 (i). The construction consists of  $4k+1$  consecutive blocks,  $X, D_1, G_1, D_2, G_2, \dots, G_{2k-1}, D_{2k}, Y$ , in this order (from bottom to top). Let  $m = 2 \cdot 4^k$ . We define  $2km$  independent edges  $\alpha_i^j$  and  $\beta_i^j$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq m$ , and two isolated vertices  $x$  and  $y$  as follows. Put  $x$  in  $X$  and  $y$  in  $Y$ .

- The edges  $\alpha_1^j$  and  $\beta_1^j$ ,  $1 \leq j \leq m$ , are in  $D_1$  and  $G_1$ .
- For every  $i$ ,  $1 < i < 2k$ , the edges  $\alpha_i^j$  and  $\beta_i^j$ ,  $1 \leq j \leq m$ , are in  $G_{i-1}$ ,  $D_i$  and  $G_i$ .
- The edges  $\alpha_{2k}^j$  and  $\beta_{2k}^j$ ,  $1 \leq j \leq m$ , are in  $G_{2k-1}$  and  $D_{2k}$ .

For every  $i$ ,  $1 \leq i \leq 2k$ ,  $D_i$  is a double- $k$ -forcing block, and the edges  $\alpha_i^j$ ,  $1 \leq j \leq m$ , and  $\beta_i^j$ ,  $1 \leq j \leq m$ , form its two groups  $D'$  and  $D''$ . For every  $i$ ,  $1 \leq i \leq 2k-1$ ,  $G_i$  is a  $(2, k)$ -grid block  $G(2, k)$  with groups of cables  $G'$  and  $G''$ . The edges  $\alpha_i^j$  form a cable  $G'_1$ , the edges  $\beta_i^j$  form a cable  $G'_2$ , the edges  $\alpha_{i+1}^j$  form a cable  $G''_1$ , and the edges  $\beta_{i+1}^j$  form a cable  $G''_2$ . Cables  $G'_1$  and  $G'_2$  form the group  $G'$ , and cables  $G''_1$  and  $G''_2$  form the group  $G''$ .

Let  $\kappa$  be a fixed curve from  $x$  to  $y$ . We show that  $\kappa$  crosses one of the curves  $\alpha_i^j$  or  $\beta_i^j$  at least  $2k$  times.

For every  $i$ ,  $1 \leq i \leq 2k$ , by Observation 11, there is a  $j$ ,  $1 \leq j \leq m$ , such that  $\alpha_i^j$  and  $\kappa$  cross at least  $k$  times in  $D_i$ . For simplicity, denote this  $\alpha_i^j$  by  $\alpha_i$ . Similarly, there is a  $j'$ ,  $1 \leq j' \leq m$ , such that  $\beta_i^{j'}$  and  $\kappa$  cross at least  $k$  times in  $D_i$ . Denote  $\beta_i^{j'}$  by  $\beta_i$ .

For every  $Z \in \{D_1, D_2, \dots, D_{2k}, G_1, G_2, \dots, G_{2k-1}\}$  and  $\chi \in \{\alpha_1, \dots, \alpha_{2k}, \beta_1, \dots, \beta_{2k}\}$ , let  $Z(\chi)$  denote the number of intersections of  $\chi$  with  $\kappa$  in  $Z$ . By the choice of  $\alpha_i$  and  $\beta_i$ , for every  $i$ ,  $1 \leq i \leq 2k$ , we have

$$D_i(\alpha_i), D_i(\beta_i) \geq k.$$

By Observation 10, for every  $i$ ,  $1 \leq i \leq 2k-1$ , we have

$$G_i(\alpha_i) + G_i(\beta_i) + G_i(\alpha_{i+1}) + G_i(\beta_{i+1}) \geq 2k-1.$$

Summing up,

$$\begin{aligned} & \sum_{i=1}^{2k} (\text{CR}(\kappa, \alpha_i) + \text{CR}(\kappa, \beta_i)) \\ &= \sum_{i=1}^{2k} (D_i(\alpha_i) + D_i(\beta_i)) + \sum_{i=1}^{2k-1} (G_i(\alpha_i) + G_i(\alpha_{i+1}) + G_i(\beta_i) + G_i(\beta_{i+1})) \\ &\geq 4k^2 + (2k-1)^2 = 4k(2k-1) + 1. \end{aligned}$$

Therefore, for some  $i$ ,  $\text{CR}(\kappa, \alpha_i) \geq 2k$  or  $\text{CR}(\kappa, \beta_i) \geq 2k$ . □

### 3 Proof of Theorem 2 (ii)

Let  $G$  be a non-complete  $k$ -simple topological graph, and let  $u$  and  $v$  be two non-adjacent vertices of  $G$ . We prove that  $u$  and  $v$  can be connected by a curve that has at most  $2k$  points in common with any edge of  $G$ .

Place a new vertex at each crossing of  $G$  and subdivide the edges accordingly. Let  $G'$  denote the resulting topological (multi)graph. Clearly, there is no loss of generality in assuming that  $G'$  is connected. Choose an arbitrary path  $\alpha$  in  $G'$  connecting  $u$  and  $v$ . We distinguish two types of vertices on  $\alpha$ . A vertex  $x$  of  $G'$  that lies on  $\alpha$  is called a *passing vertex* if the two edges of  $\alpha$  incident to  $x$  belong to the same edge of  $G$ . A vertex  $x$  of  $G'$  that lies on  $\alpha$  is a *turning vertex* if it is not a passing vertex, that is, if the two edges of  $\alpha$  meeting at  $x$  belong to distinct edges of  $G$ .

Assign to  $\alpha$  a unique *code*, denoted by  $c(\alpha)$ , as follows. Suppose that  $\alpha$  contains  $r$  turning vertices for some  $r \geq 0$ . These vertices divide  $\alpha$  into  $r + 1$  intervals,  $I_1^\alpha, I_2^\alpha, \dots, I_{r+1}^\alpha$ , ordered from  $u$  to  $v$ . Set  $p_0^\alpha = r$  and for any  $i$ ,  $1 \leq i \leq r + 1$ , let  $p_i^\alpha$  denote the number of passing vertices on  $I_i^\alpha$ . Let  $c(\alpha) = (p_0^\alpha, p_1^\alpha, p_2^\alpha, \dots, p_{r+1}^\alpha)$ ; see Figure 7.

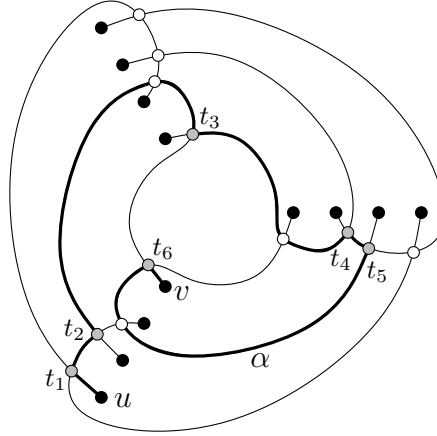


Figure 7: A  $(u, v)$ -path  $\alpha$  (in bold) with  $c(\alpha) = (6, 0, 0, 1, 1, 0, 1, 0)$  and its turning vertices  $t_i$ .

Order the codes of all  $(u, v)$ -paths lexicographically: if  $\alpha$  and  $\beta$  are two  $(u, v)$ -paths in  $G'$ , with codes  $c(\alpha) = (p_0^\alpha = r, p_1^\alpha, p_2^\alpha, \dots, p_{r+1}^\alpha)$  and  $c(\beta) = (p_0^\beta = s, p_1^\beta, p_2^\beta, \dots, p_{s+1}^\beta)$ , respectively, then let  $c(\alpha) \prec_{\text{lex}} c(\beta)$  if and only if  $c(\alpha) \neq c(\beta)$  and for the smallest index  $i$  such that  $p_i \neq q_i$  we have  $p_i < q_i$ .

Finally, define a *partial ordering*  $\prec$  on the set of all the  $(u, v)$ -paths in  $G'$ : for any two  $(u, v)$ -paths,  $\alpha$  and  $\beta$ , let  $\alpha \prec \beta$  if and only if  $c(\alpha) \prec_{\text{lex}} c(\beta)$ .

Let  $\gamma$  be a *minimal* element with respect to  $\prec$ . Suppose that  $\gamma$  has  $r$  turning vertices,  $t_1, t_2, \dots, t_r$ ,  $r \geq 0$ , which divide  $\gamma$  into intervals  $I_1^\gamma, I_2^\gamma, \dots, I_{r+1}^\gamma$ , ordered from  $u$  to  $v$ . Consider the intervals as half-closed, that is, for every  $i$ ,  $0 \leq i \leq r$ , let  $t_i$  belong to  $I_{i+1}^\gamma$ .

Next we establish some simple properties of the intersections of  $\gamma$  with the edges of  $G$ .

**Lemma 12.** *Let  $e$  be an edge of  $G$  that has only finitely many points in common with  $\gamma$ . Then all of these points belong to two consecutive intervals of  $\gamma$ .*

*Proof.* Suppose for contradiction that  $e$  has nonempty intersection with at least two non-consecutive intervals of  $\gamma$ . Let  $x$  (and  $y$ ) denote the crossing of  $e$  and  $\gamma$ , closest to (respectively,

farthest from)  $u$  along  $\gamma$ . Let  $x$  belong to  $I_i^\gamma$  and let  $y$  belong to  $I_j^\gamma$ , where  $i < j - 1$ .

Let  $\gamma'$  be another  $(u, v)$ -path, which is identical to  $\gamma$  from  $u$  to  $x$ , identical to  $e$  from  $x$  to  $y$ , and finally identical to  $\gamma$  from  $y$  to  $v$ ; see Figure 8. If  $i < j - 2$ , then it is evident that  $c(\gamma') \prec_{\text{lex}} c(\gamma)$ , since  $\gamma'$  has fewer turning vertices than  $\gamma$ . If  $i = j - 2$ , then  $\gamma$  and  $\gamma'$  have the same number of turning vertices, but  $I_i^{\gamma'}$  contains fewer passing vertices than  $I_i^\gamma$  (hence  $p_i^{\gamma'} < p_i^\gamma$ ), and we have  $c(\gamma') \prec_{\text{lex}} c(\gamma)$ . In both cases we obtain that  $\gamma' \prec \gamma$ , contradicting the minimality of  $\gamma$ .  $\square$

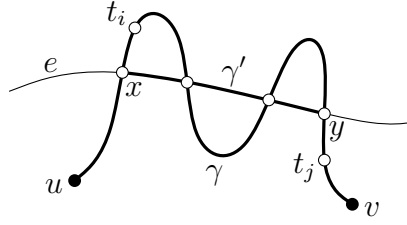


Figure 8: Two  $(u, v)$ -paths  $\gamma$  and  $\gamma'$  (both in bold) in the proof of Lemma 12.

**Lemma 13.** *Let  $e$  be an edge of  $G$  that has only finitely many points in common with  $\gamma$ .*

- (i) *If none of the common points is a vertex of  $e$ , then  $e$  crosses  $\gamma$  at most  $2k$  times.*
- (ii) *If one of the common points is a vertex of  $e$ , then  $e$  crosses  $\gamma$  at most  $2k - 1$  times.*

*Proof.* First, suppose that no vertex of  $e$  lies on  $\gamma$ . By Lemma 12,  $e$  crosses at most two consecutive intervals of  $\gamma$ . Each interval is a part of some edge of  $G$  and hence crosses  $e$  at most  $k$  times. This proves (i).

Suppose next that one of the vertices of  $e$  lies on  $\gamma$ . Observe that such a vertex must be a turning vertex of  $\gamma$ , say  $t_i$ . Again, by Lemma 12,  $e$  crosses at most two consecutive intervals of  $\gamma$ . Each interval is a part of some edge of  $G$ . Moreover, one of them has a common endpoint with  $e$ . Therefore,  $e$  crosses one of the intervals at most  $k$  times and the other at most  $k - 1$  times. This proves (ii).  $\square$

Note that no edge  $e$  of  $G$  that has only finitely many points in common with  $\gamma$  can have both of its endpoints on  $\gamma$ . Otherwise, both endpoints must be turning vertices of  $\gamma$ , say  $t_i$  and  $t_j$  for some  $i < j$ . Since the underlying abstract graph  $G$  is simple (that is,  $G$  has no multiple edges), the edge of  $G$  that contains  $I_{i+1}^\gamma$  must be different from the edge that contains  $I_j^\gamma$ . Hence, there is at least one turning vertex between  $t_i$  and  $t_j$  on  $\gamma$ . Now consider another  $(u, v)$ -path  $\gamma'$  that is identical to  $\gamma$  from  $u$  to  $t_i$ , identical to  $e$  from  $t_i$  to  $t_j$ , and finally identical to  $\gamma$  from  $t_j$  to  $v$ . The turning vertices  $t_i$  and  $t_j$  of  $\gamma$  are also turning vertices on  $\gamma'$ . Since the turning vertices of  $\gamma$  that lie between  $t_i$  and  $t_j$  are not among the turning vertices of  $\gamma'$ ,  $\gamma'$  has fewer turning vertices than  $\gamma$ . Therefore, we have  $c(\gamma') \prec_{\text{lex}} c(\gamma)$ , contradicting the minimality of  $\gamma$ .

**Lemma 14.** *Let  $e$  be an edge of  $G$  that contains an interval  $I_i^\gamma$  of  $\gamma$ . Then  $e$  and  $\gamma$  have at most  $k$  points in common outside of  $I_i^\gamma$ . Furthermore, one of these points is  $t_i$ , the endpoint of  $I_i^\gamma$ .*

*Proof.* Since  $I_i^\gamma$  and  $I_{i+1}^\gamma$  are separated by  $t_i$ , and  $I_i^\gamma$  is contained in  $e$ , it follows that  $e$  cannot contain  $I_{i+1}^\gamma$ . Similarly,  $e$  cannot contain  $I_{i-1}^\gamma$ .

If  $e$  has a point  $p$  in  $I_j^\gamma$  with  $j < i$ , consider another  $(u, v)$ -path  $\gamma'$  that is identical to  $\gamma$  from  $u$  to  $p$ , identical to  $e$  from  $p$  to  $t_{i-1}$ , and finally identical to  $\gamma$  from  $t_{i-1}$  to  $v$ ; see Figure 9. If  $j < i - 1$ , the turning vertices  $t_j$  and  $t_{i-1}$  of  $\gamma$  are not among the turning vertices of  $\gamma'$ . Although  $p$  was a passing vertex of  $\gamma$  and is now a turning vertex of  $\gamma'$ , still  $\gamma'$  has fewer turning vertices than  $\gamma$ . Therefore,  $c(\gamma') \prec_{\text{lex}} c(\gamma)$ . If  $j = i - 1$ , the turning vertex  $t_j$  of  $\gamma$  is not a turning vertex of  $\gamma'$ . Again,  $p$  was a passing vertex of  $\gamma$  and is now a turning vertex of  $\gamma'$ . So,  $\gamma$  and  $\gamma'$  have the same number of turning vertices. Since  $p$  is not a passing vertex of  $\gamma'$ ,  $I_{i-1}^{\gamma'}$  has fewer passing vertices than  $I_{i-1}^\gamma$  (hence  $p_{i-1}^{\gamma'} < p_{i-1}^\gamma$ ), and we have that  $c(\gamma') \prec_{\text{lex}} c(\gamma)$ . In all of the above cases, we obtain that  $\gamma' \prec \gamma$ , contradicting the minimality of  $\gamma$ .

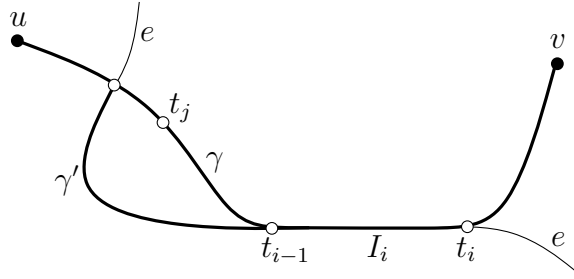


Figure 9: Two  $u, v$ -paths  $\gamma$  and  $\gamma'$  (both in bold) in the proof of Lemma 14;  $j < i - 1$ .

Similarly, if  $e$  has a point  $p$  in  $I_j^\gamma$  with  $j > i + 1$ , consider another  $(u, v)$ -path  $\gamma'$  that is identical to  $\gamma$  from  $u$  to  $t_i$ , identical to  $e$  from  $t_i$  to  $p$ , and finally identical to  $\gamma$  from  $p$  to  $v$ . The turning vertices  $t_i$  and  $t_{j-1}$  of  $\gamma$  are not among the turning vertices of  $\gamma'$ . Although  $p$  was a passing vertex of  $\gamma$  and is a turning vertex of  $\gamma'$ , still  $\gamma'$  has fewer turning vertices than  $\gamma$ . Therefore,  $c(\gamma') \prec_{\text{lex}} c(\gamma)$ , contradicting the minimality of  $\gamma$ .

Note that the case  $j = i + 1$  cannot be settled in the same way as the previous cases, since the number of passing vertices on  $e$  between  $t_i$  and  $p$  may not be smaller than the number of passing vertices on  $\gamma$  between  $t_i$  and  $p$ . Nevertheless, we can conclude that no interval of  $\gamma$  other than  $I_i^\gamma$  is contained in  $e$ . Furthermore, the only interval of  $\gamma$  other than  $I_i^\gamma$  that can share some points with  $e$  is  $I_{i+1}^\gamma$ . Let  $f$  be the edge of  $G$  that contains  $I_{i+1}^\gamma$ . Since  $e$  and  $f$  have at most  $k$  points in common,  $e$  and  $I_{i+1}^\gamma$  can have at most  $k$  points in common, too. The point  $t_i$ , the common endpoint of  $I_i^\gamma$  and  $I_{i+1}^\gamma$ , is one of these points.  $\square$

Now we are in a position to complete the proof of Theorem 2 (ii). Join  $u$  and  $v$  by a curve  $\beta$  that runs very close to  $\gamma$ .

We claim that any edge  $e$  of  $G$  has at most  $2k$  points in common with  $\beta$ . If  $e$  has only finitely many points in common with  $\gamma$  and none of them is a vertex of  $e$ , then every crossing between  $e$  and  $\beta$  corresponds to a crossing between  $e$  and  $\gamma$ . Therefore, by Lemma 13(i),  $e$  and  $\beta$  cross each other at most  $2k$  times. If  $e$  has only finitely many points in common with  $\gamma$ , but one of them is a vertex of  $e$ , then each crossing between  $e$  and  $\beta$  corresponds to a crossing between  $e$  and  $\gamma$ , and there may be an additional crossing near the vertex of  $e$  on  $\gamma$ . Again, by Lemma 13(ii), there are at most  $2k$  crossings between  $e$  and  $\beta$ . Finally, if  $e$  contains a whole interval  $I_i^\gamma$  of  $\gamma$ , then each crossing between  $e$  and  $\beta$  corresponds to a crossing between

$e$  and  $\gamma$ , or to a vertex of  $e$  on  $\gamma$ . There may be an additional crossing near the endpoint  $t_i$  of  $I_i^\gamma$ . Thus, there are at most  $k + 1$  crossings.  $\square$

## 4 Proof of Theorem 1: Upper Bounds

The construction for  $k = 1$  is essentially different from the constructions for  $k > 1$ . For  $k > 1$ , all constructions are variations of the constructions used in the proofs of Theorem 2, Lemma 6 and Lemma 9, but they give different bounds for different values of  $k$ . Table 1 shows our best upper bounds for different values of  $k$ .

**First construction.** This construction is for  $k = 1$ . First we need to modify the graph  $G_1$  on Figure 1. Consider the edges of  $G_1$  incident to  $x$ , and modify them in a small neighborhood of  $x$  so that the resulting edges have distinct endpoints, they pairwise cross each other, and their union encloses a *region*  $X$  (i.e., a connected component  $X$  of the complement of the union of the edges) which contains  $x$ . Analogously, modify the other three edges of  $G_1$  in a small neighborhood of  $y$ . Let  $Y$  be the region that contains  $y$  and is enclosed by the modified edges. The resulting simple topological graph  $G$  has 12 vertices and 6 edges; see Figure 10. The points  $x, y \in V(G_1)$  do not belong to  $V(G)$ .

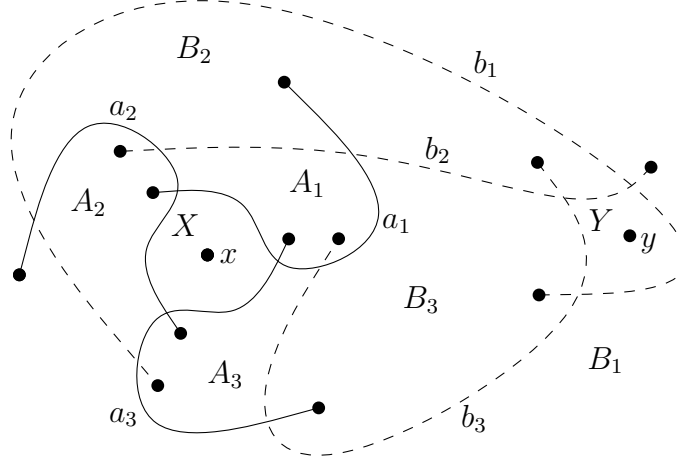


Figure 10: A topological graph  $G$ : the edge  $\{x, y\}$  cannot be added.

**Lemma 15.** *Let  $x$  and  $y$  be any pair of points belonging to the regions  $X$  and  $Y$  in  $G$ , respectively. Then any curve joining  $x$  and  $y$  will meet at least one of the edges of  $G$  at least twice.*

*Proof.* We prove the claim by contradiction. Let  $a_1, a_2, a_3, b_1, b_2, b_3$  denote the edges of  $G$ . They divide the plane into eight regions,  $X, Y, A_1, A_2, A_3, B_1, B_2, B_3$ ; see Figure 10. Suppose there exists an oriented curve from  $x$  to  $y$  that crosses every edge of  $G$  at most once. Let  $\gamma$  be such a curve with the smallest number of crossings with the edges of  $G$ . Let  $c_1, c_2, \dots, c_{m-1}$  be the crossings between  $\gamma$  and the edges of  $G$ , ordered according to the orientation of  $\gamma$ . They divide  $\gamma$  into intervals  $I_1, I_2, \dots, I_m$ , ordered again according to the orientation of  $\gamma$ . The first interval  $I_1$  lies in  $X$ , and the last one,  $I_m$ , lies in  $Y$ . Observe that no other interval

can belong to  $X$  or to  $Y$ , because in this case we could simplify  $\gamma$  and obtain a curve with a smaller number of crossings. By symmetry, we can assume that the first crossing,  $c_1$ , is a crossing between  $\gamma$  and  $a_1$ . Then  $I_2$  belongs to  $A_1$ . The following property holds.

*Property  $\mathcal{P}$ : If for some  $j \geq 2$ , the interval  $I_j$  belongs to  $A_i$  (or  $B_i$ ), then one of the points  $c_1, c_2, \dots, c_{j-1}$  is a crossing between  $\gamma$  and the edge  $a_i$  (or  $b_i$ , respectively).*

We prove Property  $\mathcal{P}$  by induction on  $j$ . Clearly, the property holds for  $j = 2$ . Assume that  $I_{j-1}$  is in  $A_i$  (or  $B_i$ ) and one of  $c_1, c_2, \dots, c_{j-2}$  is a crossing between  $\gamma$  and  $a_i$  (or  $b_i$ ). For simplicity, assume that  $I_{j-1}$  belongs to the region  $A_1$  and that one of the points  $c_1, c_2, \dots, c_{j-2}$  is a crossing between  $\gamma$  and  $a_1$ ; the other cases are analogous. Since  $c_{j-1}$  cannot belong to  $a_1$ , it must be a crossing between  $\gamma$  and either  $a_2$  or  $b_2$ . In the first case,  $I_j$  belongs to  $A_2$ , in the second to  $B_2$ . In either case, Property  $\mathcal{P}$  is preserved.

Now, we can complete the proof of Lemma 15. Consider the interval  $I_{m-1}$ . Since  $I_m$  lies in  $Y$ , for some  $i$ , the interval  $I_{m-1}$  must lie in  $B_i$ . Suppose for simplicity that  $I_{m-1}$  lies in  $B_1$ . By Property  $\mathcal{P}$  (with  $j = m - 1$ ,  $m \geq 3$ ), one of the points  $c_1, c_2, \dots, c_{m-2}$  must be a crossing between  $\gamma$  and  $b_1$ . However, using that  $I_m$  is in  $Y$ ,  $c_{m-1}$  must be another crossing between  $\gamma$  and  $b_1$ . Thus,  $\gamma$  crosses  $b_1$  twice, which is a contradiction.  $\square$

Now, we return to the proof of the upper bound in Theorem 1. Modify the drawing of  $G$  in Figure 10 so that the region  $Y$  becomes unbounded, and let  $H$  be the resulting topological graph. Denote by  $Y$  the *outer region* of  $H$  and by  $X$  the *inner region* of  $H$ ; see Figure 11.

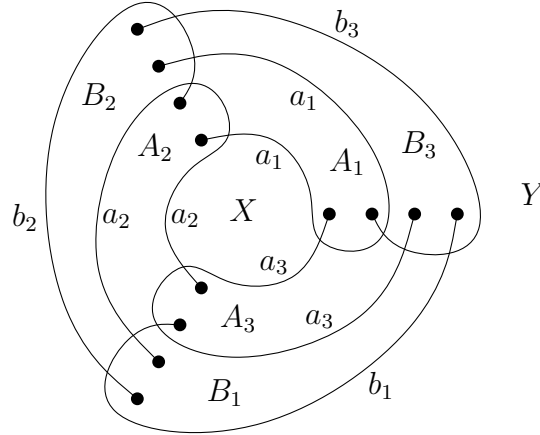


Figure 11: A topological graph  $H$ , a modification of  $G$ .

For every  $n \geq 1$ , construct a saturated simple topological graph  $F_n$ , as follows. Let  $k = \lfloor n/12 \rfloor$ . Take a disjoint union of  $k$  scaled and translated copies of  $H$ , denoted by  $H^1, H^2, \dots, H^k$ , such that for any  $i$ ,  $1 < i \leq k$ , the copy  $H^i$  lies entirely in the inner region of  $H^{i-1}$ ; see Figure 12. For  $1 \leq i \leq k$ , let  $V_i$  be the vertex set of  $H^i$ . Finally, place  $n - 12k$  additional vertices in the inner region of  $H^k$ , and let  $V_{k+1}$  denote the set of these vertices. Obviously, we have  $|V_{k+1}| < 12$ .

Add to this topological graph all possible missing edges one by one, in an arbitrary order, as long as it remains simple. We end up with a saturated simple topological graph  $F_n$  with  $n$  vertices. Observe that for every  $i$  and  $j$  with  $1 \leq i < j - 1 < k$ ,  $V_i$  lies in the outer region of  $H^{i+1}$ , while  $V_j$  is in the inner region of  $H^{i+1}$ . By Lemma 15 (applied with  $G = H^{i+1}$ ,  $x \in V_j$ ,



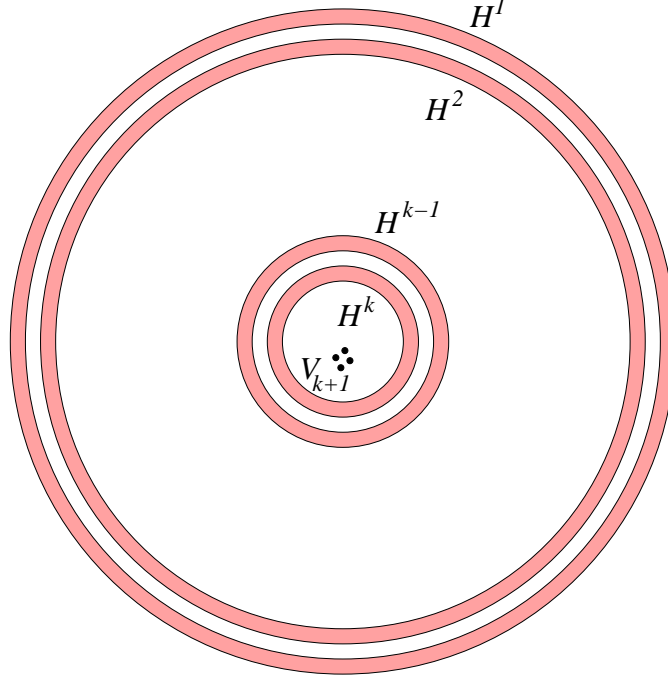


Figure 12: A saturated simple topological graph  $F_n$ .

$y \in V_i$ ), no edge of  $F_n$  runs between  $V_i$  and  $V_j$ . Hence, every vertex in  $V_i$  can be adjacent to at most 35 other vertices; namely, to the elements of  $V_{i-1} \cup V_i \cup V_{i+1}$ . Therefore,  $F_n$  is a saturated simple topological graph with  $n$  vertices and at most  $17.5n$  edges.

**Second construction.** This construction is used for all odd  $k \geq 5$  and all even  $k \geq 12$ . Suppose for simplicity that  $n$  is divisible by 3 and let  $m = n/3$ . The construction consists of  $2m + 3$  consecutive blocks,  $B_0, A_0, B_1, A_1, \dots, B_m, A_m, B_{m+1}$ , in this order, from bottom to top. See Figure 13, left.

For every  $i$ ,  $1 \leq i \leq m$ , let  $u_i$  be a vertex on the common boundary of  $B_{i-1}$  and  $A_{i-1}$  and let  $v_i$  and  $w_i$  be vertices on the common boundary of  $A_i$  and  $B_{i+1}$ . Let  $\alpha_i$  be an edge connecting  $u_i$  and  $v_i$  and let  $\beta_i$  be an edge connecting  $u_i$  and  $w_i$ . The pair  $(\alpha_i, \beta_i)$  is called the  $i$ th *bundle*. The edges  $\alpha_i$  and  $\beta_i$  form a  $(k-1)$ -spiral in  $B_i$ . For  $1 < i \leq m$ , the edges  $\alpha_i$  and  $\beta_i$  form a cable in  $A_{i-1}$ , the edges  $\alpha_{i-1}$  and  $\beta_{i-1}$  also form a cable in  $A_{i-1}$ , and these two cables form a  $k$ -spiral in  $A_{i-1}$ . The resulting  $k$ -simple topological graph  $G$  has  $n$  vertices and  $2n/3$  edges. Add to  $G$  all possible missing edges one by one, as long as the drawing remains  $k$ -simple. We obtain a saturated  $k$ -simple topological graph  $H$ . Note that  $H$  is not uniquely determined by  $G$ , not even as an abstract graph.

Suppose that  $k \geq 11$ . Just like in the proof of Lemma 6, we can prove that any curve from  $A_i$  to  $A_{i+3}$  has to cross one of the curves  $\alpha_{i+1}, \beta_{i+1}, \alpha_{i+2}, \beta_{i+2}, \alpha_{i+3}, \beta_{i+3}$  at least  $(k-2)/2 + 2(k-1)/3 > k$  times. Therefore, in  $H$ , a vertex from the block  $A_i$  can possibly be connected only to other vertices from the five blocks  $A_{i-2}, A_{i-1}, A_i, A_{i+1}, A_{i+2}$ . Since every block  $A_j$  has at most three vertices, the maximum degree in  $H$  is at most  $5 \cdot 3 - 1 = 14$  and thus  $H$  has at most  $7n$  edges.

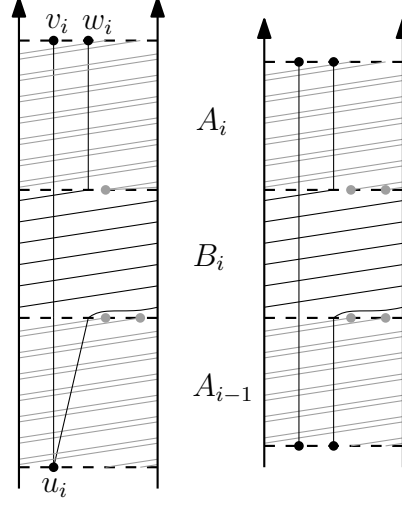


Figure 13: Three consecutive blocks in the constructions of saturated  $k$ -simple topological graphs. Left: the second construction for  $k = 7$ . Right: the third construction for  $k = 6$ .

For  $k \geq 9$  odd and for every  $j$ ,  $1 \leq j \leq 3$ , every curve  $\kappa$  from  $A_i$  to  $A_{i+3}$  has to cross one of the two curves  $\alpha_{i+j}, \beta_{i+j}$  at least  $(k-1)/2$  times in  $B_{i+j}$ . Let  $\gamma_{i+j}$  be this curve. Now, for every  $j$ ,  $1 \leq j \leq 2$ , the curve  $\kappa$  crosses  $\gamma_{i+j}$  and  $\gamma_{i+j+1}$  together at least  $k-1$  times in  $A_{i+j}$ . It follows that  $\kappa$  crosses one of the curves  $\gamma_{i+1}, \gamma_{i+2}, \gamma_{i+3}$  at least  $(1/2 + 2/3) \cdot (k-1) > k$  times.

Therefore, in  $H$ , a vertex from the block  $A_i$  can be connected only to other vertices from the five blocks  $A_{i-2}, A_{i-1}, \dots, A_{i+2}$ . The maximum degree in  $H$  is thus at most  $5 \cdot 3 - 1 = 14$  and  $H$  has at most  $7n$  edges.

Similarly, for  $k \geq 7$  odd, for every curve  $\kappa$  from  $A_i$  to  $A_{i+4}$ , there are four curves  $\gamma_{i+1}, \gamma_{i+2}, \gamma_{i+3}, \gamma_{i+4}$  such that  $\kappa$  crosses one of them at least  $(1/2 + 3/4) \cdot (k-1) > k$  times.

Therefore, in  $H$ , a vertex from the block  $A_i$  can be connected only to vertices from the seven blocks  $A_{i-3}, A_{i-2}, \dots, A_{i+3}$ . The maximum degree in  $H$  is thus at most  $7 \cdot 3 - 1 = 20$  and  $H$  has at most  $10n$  edges.

For  $k \geq 5$  odd, for every curve  $\kappa$  from  $A_i$  to  $A_{i+5}$ , there are five curves,  $\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{i+5}$  such that  $\kappa$  crosses one of them at least  $(1/2 + 4/5) \cdot (k-1) > k$  times.

Therefore, in  $H$ , a vertex from the block  $A_i$  can be connected only to vertices from the nine blocks  $A_{i-4}, A_{i-3}, \dots, A_{i+4}$ . The maximum degree in  $H$  is thus at most  $9 \cdot 3 - 1 = 26$  and  $H$  has at most  $13n$  edges.

**Third construction.** This construction is used for  $k \in \{4, 6, 8, 10\}$ . It is a modification of the second construction, where the edges of the  $i$ th bundle,  $\alpha_i$  and  $\beta_i$ , do not have common endpoints, so they form a matching rather than a path, and they form a  $k$ -spiral in  $B_i$ . See Figure 13, right.

For  $k \geq 6$  even and for every  $j$ ,  $1 \leq j \leq 3$ , every curve  $\kappa$  from  $A_i$  to  $A_{i+3}$  has to cross one of the two curves  $\alpha_{i+j}, \beta_{i+j}$  at least  $k/2$  times in  $B_{i+j}$ . Let  $\gamma_{i+j}$  be this curve. Now, for every  $j$ ,  $1 \leq j \leq 2$ , the curve  $\kappa$  crosses  $\gamma_{i+j}$  and  $\gamma_{i+j+1}$  together at least  $k-1$  times in  $A_{i+j}$ . It follows that  $\kappa$  crosses one of the curves  $\gamma_{i+1}, \gamma_{i+2}, \gamma_{i+3}$  at least  $k/2 + 2(k-1)/3 > k$  times.

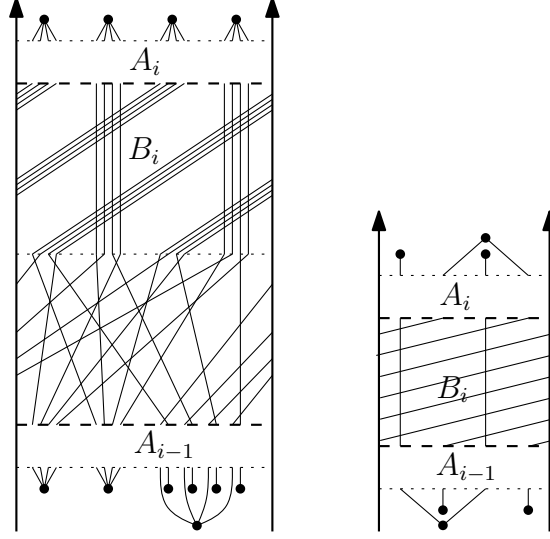


Figure 14: A bundle in the construction of a saturated 2-simple (left) and a saturated 3-simple (right) topological graph.

Therefore, in  $H$ , a vertex from the block  $A_i$  can be connected only to other vertices from the five blocks  $A_{i-2}, A_{i-1}, \dots, A_{i+2}$ . Since every block  $A_j$  now has at most four vertices, the maximum degree in  $H$  is at most  $5 \cdot 4 - 1 = 19$  and  $H$  has at most  $9.5n$  edges.

Similarly for  $k \geq 4$  even, for every curve  $\kappa$  from  $A_i$  to  $A_{i+4}$ , there are four curves  $\gamma_{i+1}, \gamma_{i+2}, \gamma_{i+3}, \gamma_{i+4}$  such that  $\kappa$  crosses one of them at least  $k/2 + 3(k-1)/4 > k$  times.

Therefore, in  $H$ , a vertex from the block  $A_i$  can be connected only to other vertices from the seven blocks  $A_{i-3}, A_{i-2}, \dots, A_{i+3}$ . The maximum degree in  $H$  is thus at most  $7 \cdot 4 - 1 = 27$  and  $H$  has at most  $13.5n$  edges.

**Fourth construction.** This construction is for  $k = 2$ . First we present a weaker but simpler version. It is a modification of the previous constructions. Here each bundle contains 16 independent edges. The edges of the  $i$ th bundle form a 2-forcing block in  $B_i$ . In  $A_i$ , the edges of the  $i$ th bundle form a cable, the edges of the  $(i+1)$ st bundle form another cable, and these two cables form a 2-spiral. Let  $\kappa$  be a curve from  $A_i$  to  $A_{i+2}$ . Just like in the previous arguments, using Observation 5 and Lemma 8, it is not hard to see that  $\kappa$  has to cross an edge more than twice. Therefore, in  $H$ , a vertex from  $A_i$  can be connected only to vertices from  $A_{i-1}, A_i, A_{i+1}$ . Every block  $A_j$  has at most 32 vertices, so the maximum degree in  $H$  is at most 95, therefore,  $H$  has at most  $47.5n$  edges.

The best construction we have is very similar. To obtain it, in each bundle we identify some of the endpoints of the edges, and we also modify the order of the edges along the bottom boundary of  $B_i$ ; see Figure 14, left. Then every block  $A_i$  has at most 11 vertices, so the maximum degree in  $H$  is at most  $3 \cdot 11 - 1 = 32$  and  $H$  has at most  $16n$  edges.

**Fifth construction.** This construction is for  $k = 3$ . First we present a weaker but simpler version. It is again a modification of the previous constructions. Here each bundle contains four independent edges. The edges of the  $i$ th bundle form a grid block  $G(2, 3)$  in  $B_i$ . In  $A_i$ ,

the edges of the  $i$ th bundle form a cable, the edges of the  $(i+1)$ st bundle form another cable, and these two cables form a 3-spiral. Let  $\kappa$  be a curve from  $A_i$  to  $A_{i+3}$ . Just like in the previous arguments, using Observations 10 and 5, it is not hard to see that  $\kappa$  has to cross an edge more than three times. Therefore, in  $H$ , a vertex from  $A_i$  can be connected only to vertices from  $A_{i-2}, A_{i-1}, \dots, A_{i+2}$ . Every block  $A_j$  has at most 8 vertices, so  $H$  has at most  $19.5n$  edges.

To obtain our best construction, in each bundle we identify some endpoints of the edges; see Figure 14, right. Then every block  $A_i$  has at most 6 vertices, so the maximum degree in  $H$  is at most  $5 \cdot 6 - 1 = 29$  and  $H$  has at most  $14.5n$  edges. A modification of this construction works for  $k = 1$ , and it gives the same upper bound,  $17.5n$ , as the first construction.

This concludes the proof of the upper bounds.  $\square$

## 5 Proof of Theorem 1: Lower Bounds

A vertex of a (topological) graph is *isolated* if its degree is zero. A triangle in a (topological) graph is called *isolated* if its vertices are incident to no edges other than the edges of the triangle.

**Lemma 16.** *A saturated simple topological graph on at least four vertices contains no isolated triangle.*

*Proof.* Let  $G$  be a saturated simple topological graph with at least four vertices, and suppose for contradiction that  $G$  has an isolated triangle  $T$  with vertices  $x, y$  and  $z$ . By definition, the edges of  $T$  do not cross one another.

If all vertices other than  $x, y, z$  are isolated, it is trivial to add a new edge without crossings. Hence we may assume that  $G$  has an edge not contained in  $T$ . We distinguish two cases.

**Case 1.** The edges of  $T$  cross no other edges.

The edges of  $G$  divide the plane into regions. Let  $R$  denote a region bounded by the edges of  $T$  and at least one other nontrivial curve  $\omega$ . Let  $e = \{u, v\}$  be an edge that contributes to  $\omega$ , and let  $p$  be a point on  $e$  that belongs to the boundary of  $R$ ; see Figure 15, left. Choose a point  $p'$  inside of  $R$ , very close to  $p$ . Let  $\beta$  be a curve running inside  $R$  that connects a vertex of  $T$ , say  $x$ , to  $p'$ . Let  $\beta'$  be a curve joining  $p'$  and  $u$ , and running very close to the edge  $e$ . Adjoining  $\beta$  and  $\beta'$  at  $p'$ , we obtain a curve  $\gamma$  connecting  $x$  and  $u$ , two previously non-adjacent vertices of  $G$ . The curve  $\gamma$  crosses neither an edge of  $T$  or an edge of  $G$  incident to  $u$ . Since  $\beta$  is crossing-free, all crossings between  $\gamma$  and the edges of  $G$  must lie on  $\beta'$  and, hence, must correspond to crossings along the edge  $e$ . Therefore, every edge of  $G$  can cross  $\gamma$  at most once. Consequently,  $\gamma$  can be added to  $G$  as an extra edge so that the topological graph remains simple. This contradicts the assumption that  $G$  was saturated.

**Case 2.** At least one edge of  $T$  participates in a crossing.

Assume without loss of generality that  $e = \{x, y\}$  is crossed by another edge of  $G$ . Let  $p$  denote the crossing on  $e$  closest to  $x$ , and suppose that  $p$  is a crossing between  $e$  and another edge  $f = \{u, v\}$ ; see Figure 15, right. The point  $p$  divides  $f$  into two parts. At least one of them, say,  $up$ , does not cross the edge  $\{x, z\}$  of  $T$ . The edges  $e$  and  $f$  divide a small neighborhood of  $p$  into four parts. Choose a point  $p'$  in the part bounded by  $up$  and  $xp$ . Let  $\beta$  be a curve connecting  $x$  and  $p'$ , running very close to  $e$ . Let  $\beta'$  be a curve between  $p'$  and  $u$ , running very close to  $f$ . Adjoining  $\beta$  and  $\beta'$  at  $p'$  we obtain a curve  $\gamma$  connecting  $x$  and

$u$ , two vertices that were not adjacent in  $G$ . Just like in the previous case, add  $\gamma$  to  $G$  as an extra edge. The curve  $\gamma$  crosses no edge incident to  $x$  or  $u$ . Since the portion  $xp$  of  $e$  is crossing-free,  $\beta$  must be crossing-free, too. Therefore, all possible crossings between  $\gamma$  and the edges of  $G$  must lie on  $\beta'$  and, hence, correspond to crossings along  $f$ . Thus, every edge of  $G$  crosses  $\gamma$  at most once, contradicting our assumption that  $G$  was saturated.  $\square$

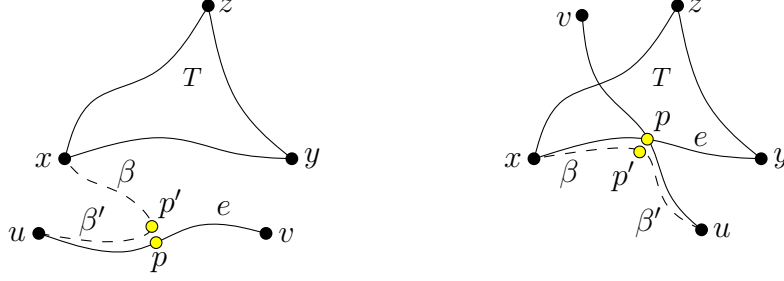


Figure 15: Case 1 and Case 2 of Lemma 16.

**Lemma 17.** *For any  $k > 0$ , a saturated  $k$ -simple topological graph on at least three vertices contains*

- (i) *no isolated vertex,*
- (ii) *no vertex of degree one.*

The proof of Lemma 17 is very similar to the proof of Lemma 16, but much easier. We omit the details.

The lower bound in Theorem 1 (ii) now follows directly. In a saturated  $k$ -simple topological graph on  $n$  vertices, every vertex has degree at least two, therefore, it has at least  $n$  edges. We are left with the proof of the lower bound of part (i). It follows immediately from the statement below.

**Lemma 18.** *In every saturated simple topological graph with at least four vertices, every vertex has degree at least 3.*

*Proof.* We prove the claim by contradiction. Let  $G$  be a saturated simple topological graph, and let  $x$  be a vertex of degree two in  $G$ . (By Lemma 17, the degree of  $x$  cannot be 0 or 1.) Let  $y$  and  $z$  denote the neighbors of  $x$ . By definition, the edges  $\{x, y\}$  and  $\{x, z\}$  do not cross. We distinguish two cases.

**Case 1.** The edges  $\{x, y\}$  and  $\{x, z\}$  cross no other edges.

By Lemma 16 and 17,  $y$  and  $z$  both have degree at least two, and  $x, y$  and  $z$  do not span an isolated triangle. Hence, at least one of the vertices  $y$  and  $z$ , say,  $y$ , has a neighbor  $w$  different from  $x$  and  $z$ . Let  $\gamma$  be a curve connecting  $x$  to  $w$  that runs very close to the edge  $\{x, y\}$  from  $x$  to a point in a small neighborhood of  $y$ , and from that point all the way to  $w$  very close to the edge  $\{y, w\}$ . We can assume that  $\gamma$  does not cross  $\{x, y\}$  and  $\{y, w\}$ . Add  $\gamma$  to  $G$  as an extra edge. Clearly,  $\gamma$  crosses no edge incident to  $x$  or  $w$ , and crosses no edge of  $G$  twice. This contradicts the assumption that  $G$  was saturated.

**Case 2.** At least one of the edges  $\{x, y\}$  and  $\{x, z\}$  participates in a crossing.

Assume without loss of generality that  $e = \{x, y\}$  is crossed by another edge of  $G$ . Let  $p$  be the crossing on  $e$  closest to  $x$ , and suppose that the other edge passing through  $p$  is  $f = \{u, v\}$ . The point  $p$  divides  $f$  into two pieces, at least one of which, say,  $up$ , has no point in common with the edge  $\{x, z\}$ . Let  $\gamma$  be a curve connecting  $x$  and  $u$ , following  $e$  very closely from  $x$  to a point in a small neighborhood of  $p$ , and from that point following  $f$  all the way to  $u$ . We can assume that  $\gamma$  does not cross  $e$  and  $f$ . Add  $\gamma$  to  $G$  as an extra edge. It is again easy to see that this new edge meets no original edge of  $G$  more than once, and again, this contradicts the assumption that  $G$  was saturated.  $\square$

## 6 Proof of Theorem 3

We start with a piece of the construction we used in the proof of Theorem 2 (i). Then we add some edges so that it remains a  $k$ -simple topological graph, and we show that it is  $(k, \lceil 3k/2 \rceil)$ -saturated.

Let  $D_1, G, D_2$  be three consecutive blocks, say, from bottom to top, and let  $m = 4^k$ . We define  $4m$  independent edges  $\alpha_i^j, \beta_i^j$ ,  $1 \leq i \leq 2$ ,  $1 \leq j \leq m$ . The edges  $\alpha_1^j$  and  $\beta_1^j$ ,  $1 \leq j \leq m$ , are in  $D_1$  and  $G$ , with endpoints on the lower boundary of  $D_1$  and the upper boundary of  $G$ . Denote the sets of these vertices by  $V_0$  and  $V_2$ , respectively. The edges  $\alpha_2^j$  and  $\beta_2^j$ ,  $1 \leq j \leq m$ , are in  $G$  and  $D_2$ , with endpoints on the lower boundary of  $G$  and the upper boundary of  $D_2$ . Denote the sets of these vertices by  $V_1$  and  $V_3$ , respectively.

For  $i = 1, 2$ , the block  $D_i$  is a double- $k$ -forcing block, the edges  $\alpha_i^j$ ,  $1 \leq j \leq m$ , and  $\beta_i^j$ ,  $1 \leq j \leq m$ , form its two groups  $D'$  and  $D''$ . The block  $G$  is a  $(2, k)$ -grid block  $G(2, k)$  with groups of cables  $G'$  and  $G''$ . The edges  $\alpha_1^j$  form a cable  $G'_1$ , the edges  $\beta_1^j$  form a cable  $G'_2$ , the edges  $\alpha_2^j$  form a cable  $G''_1$ , and the edges  $\beta_2^j$  form a cable  $G''_2$ . The cables  $G'_1$  and  $G'_2$  form the group  $G'$ , and the cables  $G''_1$  and  $G''_2$  form the group  $G''$  in  $G$ . Let  $T$  denote the resulting topological graph.

Let  $v_0 \in V_0$  and  $v_3 \in V_3$  be arbitrary vertices, and let  $\kappa$  be a curve connecting  $v_0$  and  $v_3$ . By Observation 11 there are edges  $\alpha_1 = \alpha_1^j$  and  $\beta_1 = \beta_1^{j'}$  that both cross  $\kappa$  at least  $k$  times in  $D_1$ . Similarly, there are edges  $\alpha_2 = \alpha_2^j$  and  $\beta_2 = \beta_2^{j'}$  that both cross  $\kappa$  at least  $k$  times in  $D_2$ . Since  $\alpha_1, \alpha_2, \beta_1, \beta_2$  form a  $(2, k)$ -grid block in  $G$ , by Observation 10,  $\kappa$  crosses them in  $G$  together at least  $2k - 1$  times. Therefore,  $\kappa$  crosses one of the curves at least  $\lceil (6k - 1)/4 \rceil = \lceil 3k/2 \rceil$  times.

Now we show that any two vertices  $v_i \in V_i$  and  $v_j \in V_j$  with  $|i - j| \leq 2$ , can be connected so that we still have a  $k$ -simple topological graph. We only sketch the argument. By definition, block  $D_1$  is divided into  $k$  subblocks,  $C'_1, C'_2, \dots, C'_k$ , from top to bottom. Let  $v_0 \in V_0$ ,  $v_2 \in V_2$ , let  $\alpha$  be the edge of  $T$  incident with  $v_0$  and let  $\beta$  be the edge of  $T$  incident with  $v_2$ . We can assume that  $\alpha \neq \beta$ , otherwise we are done. Draw a curve  $\kappa$  from  $v_2$  very close to  $\beta$  all the way in  $G$ , and then in the subblocks  $C'_1, C'_2, \dots, C'_{k-1}$ . In the last subblock,  $C'_k$ , connect  $\kappa$  to  $v_0$  so that it crosses all edges at most once in  $C'_k$ . A straightforward but slightly technical argument shows that it is possible. For example, we can draw the cables in  $C'_k$  as in Figure 16, and then draw  $\kappa$  as the shortest line with positive slope. Repeat this procedure for all pairs  $v_0 \in V_0$ ,  $v_2 \in V_2$ . The resulting topological graph is still  $k$ -simple. We can add similarly all edges between vertices  $v_1 \in V_1$  and  $v_3 \in V_3$ . We can connect all the remaining pairs,  $v_i \in V_i$  and  $v_j \in V_j$ ,  $|i - j| \leq 1$ , in a similar, but simpler way. We obtain a  $(k, \lceil 3k/2 \rceil)$ -saturated topological graph.  $\square$

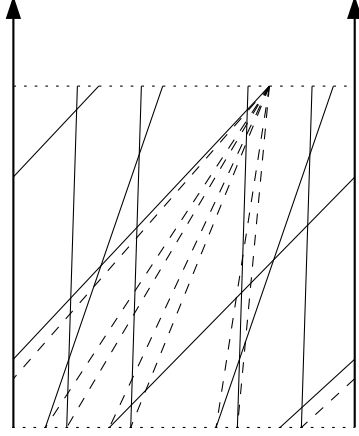


Figure 16: Adding edges to the subblock  $C_k$ .

## 7 Proof of Theorem 4

Let  $\alpha$  and  $\beta$  be two simple closed curves in the plane. Suppose that  $\alpha$  contains  $\beta$  in its interior. The region between  $\alpha$  and  $\beta$  is called an *annulus*. It is homeomorphic to the cylindrical surface, so we can transform *blocks* onto the annulus. The region outside  $\alpha$  is called the *outer exterior* of the annulus. Similarly, the region inside  $\beta$  is called the *inner exterior* of the annulus.

It is enough to prove the following statement; Theorem 4 easily follows.

**Theorem 19.** (i) For  $m \geq 2$  and  $k = 4m$ , there is a  $(3, 5k/4 - 5)$ -forcing  $k$ -pseudoline arrangement.

(ii) For  $m \geq 3$  and  $k = 2^m$ , there is a  $(k, (k/2 - 2) \cdot (\log_2 k + 1))$ -forcing  $k$ -pseudoline arrangement.

*Proof.* (i) Refer to Figure 17. First we construct an arrangement of one-way infinite curves. Let  $x_1, x_2$  and  $x_3$  be three distinct points in the plane. Let  $A_1, A_2$  and  $A_3$  be three disjoint annuli such that they contain each other in their outer exteriors, and for  $i = 1, 2, 3$ ,  $A_i$  contains  $x_i$  in its inner exterior. Let  $B$  be an annulus that contains both  $A_1$  and  $A_2$  in its inner exterior, and  $A_3$  in its outer exterior. Finally, let  $C$  be an annulus that contains both  $A_3$  and  $B$  in its inner exterior. Now we define six one-way infinite curves,  $\gamma_i^j$ , for  $i = 1, 2, 3$ ,  $j = 1, 2$ . For any fixed  $i$ ,  $i = 1, 2, 3$ , let  $\gamma_i^1$  and  $\gamma_i^2$  start very close to  $x_i$  and form a  $k/4$ -spiral in  $A_i$ . In the outer exterior of  $A_i$ , let  $\gamma_i^1$  and  $\gamma_i^2$  form a cable  $\gamma_i$ . Let  $\gamma_1$  and  $\gamma_2$  form a  $k/4$ -spiral in  $B$ . In the outer exterior of  $B$ , let  $\gamma_1$  and  $\gamma_2$  form a cable  $\gamma$ . Finally, let  $\gamma$  and  $\gamma_3$  form a  $k/4$ -spiral in  $C$ . In the outer exterior of  $C$  all six curves go to infinity.

Now replace each one-way infinite curve  $\gamma_i^j$  by two one-way infinite curves with the same endpoint, so that they go very close to each other. Each of these pairs of curves form a bi-infinite curve  $\Gamma_i^j$ , and any two intersect at most  $k$  times. For the rest of the proof we call them *pseudolines*.

Let  $\rho$  be a bi-infinite curve containing  $x_1, x_2$  and  $x_3$ . Then  $\rho$  contains at least two transversals of each of  $A_1, A_2, A_3, B$  and  $C$ . This means, by Observation 5, that in each of



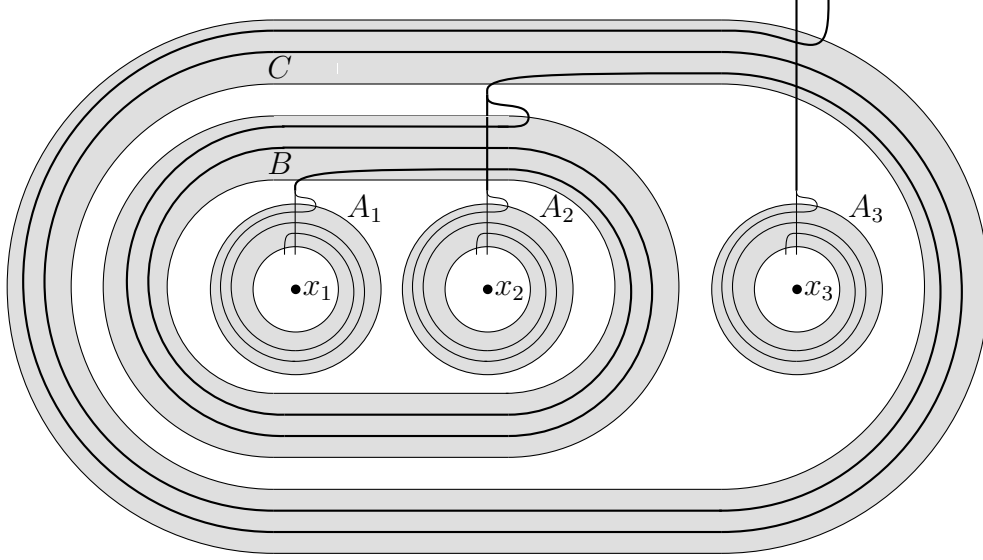


Figure 17: The arrangement from the proof of Theorem 19 (i).

$A_1$ ,  $A_2$  and  $A_3$ , there is a pseudoline, say,  $\Gamma_1^1$ ,  $\Gamma_2^1$  and  $\Gamma_3^1$ , respectively, that crosses  $\rho$  at least  $k/2 - 2$  times. Moreover,  $\rho$  crosses one of the two cables in  $B$  at least  $k/4 - 1$  times, which implies that  $\rho$  crosses one of the pseudolines  $\Gamma_1^1$  or  $\Gamma_2^1$ , say,  $\Gamma_1^1$ , at least  $k/2 - 2$  times in  $B$ . Finally,  $\rho$  crosses the two cables in  $C$  together at least  $k/2 - 2$  times. Hence, in  $C$ , the curve  $\rho$  has at least  $k/8 - 1/2$  crossings with  $\gamma$ , or at least  $3(k/8 - 1/2)$  crossings with  $\gamma_3$ . In the first case  $\rho$  crosses  $\Gamma_1^1$  at least  $5k/4 - 5$  times, in the second case it crosses  $\Gamma_3^1$  at least  $5k/4 - 5$  times.

(ii) For the second part of the theorem, we iterate the construction from the proof of part (i)  $m$  times.

Let  $P(k, 0)$  be the following arrangement. Take a point  $x$  in the plane and an annulus  $A$  around it (that is,  $x$  is in the inner exterior of  $A$ ). Let  $\gamma^1$  and  $\gamma^2$  be two one-way infinite curves, both starting near  $x$  and forming a  $k/4$ -spiral in  $A$ .

Suppose that we have already defined an arrangement  $P(k, i)$  containing  $2^i$  points and  $2^{i+1}$  one-way infinite curves. Take two disjoint copies of  $P(k, i)$ , and an annulus  $B$  that contains all annuli of both copies in its internal exterior. Merge all curves of each copy of  $P(k, i)$  into a cable and let the two cables form a  $k/4$ -spiral in  $B$ . The resulting arrangement is  $P(k, i + 1)$ .

Once the arrangement  $P(k, m)$  is constructed, take two copies of each curve in  $P(k, m)$  and join their endpoints to form a bi-infinite curve, thus obtaining a  $k$ -pseudoline arrangement  $P'(k, m)$ . Let  $X_m$  be the set of  $2^m$  points in the centers of the innermost annuli of  $P'(k, m)$ . By induction, every bi-infinite curve containing all the points of  $X_m$  crosses some pseudoline of  $P'(k, m)$  at least  $(m + 1)(k/2 - 2)$  times.  $\square$

## 8 Concluding Remarks

Our lower bound in Theorem 1 for  $k > 1$  is weaker than for  $k = 1$ . The reason is that for  $k > 1$ , we could not prove that a saturated  $k$ -simple topological graph cannot contain an isolated triangle. The main difficulty is that for  $k > 1$ , a triangle can cross itself, and our proof for Lemma 16 does not work in this case.

**Problem 1.** (i) *Is there a saturated  $k$ -simple topological graph, for some  $k \geq 2$ , that contains an isolated triangle?*

(ii) *Is there a disconnected saturated  $k$ -simple topological graph, for some  $k$ ?*

Problem 1 (ii) is open for every  $k \geq 1$ .

It follows from Theorem 2 (ii) that there is no  $(k, l)$ -saturated graph with  $l > 2k$ . By Theorem 3, there is a  $(k, l)$ -saturated graph if  $l \leq \lceil 3k/2 \rceil$ .

**Problem 2.** *Is there a  $(k, l)$ -saturated graph with  $k \geq 2$  and  $l > \lceil 3k/2 \rceil$ ?*

In Theorem 4 we have shown that for sufficiently large  $k$ , there is a  $(3, k + 1)$ -forcing arrangement of  $k$ -pseudolines. On the other hand, it is easy to see that there are no  $(1, k + 1)$ -forcing arrangements of  $k$ -pseudolines.

**Problem 3.** *Is there a  $(2, k + 1)$ -forcing arrangement of  $k$ -pseudolines for some  $k \geq 3$ ?*

We assumed that in a  $k$ -simple topological graph, no edge can cross itself. For any  $k$ , a graph drawn in the plane is called a  *$k$ -complicated topological graph* if any two edges have at most  $k$  points in common, and an edge is allowed to cross itself, at most  $k$  times. Somewhat surprisingly, for saturated  $k$ -complicated topological graphs we cannot even prove Lemma 17 part (ii). We can only prove that a saturated  $k$ -complicated topological graph does not have isolated vertices. Therefore, the best lower bound we have for the minimum number of edges of a saturated  $k$ -complicated topological graph is  $c_k(n) \geq n/2$ . On the other hand, for  $k \geq 6$ , using self-crossings, we can improve our upper bound constructions from the proof of Theorem 1 to obtain that  $c_k(n) \leq 5n/2$ . We sketch the construction here.

Suppose that  $n$  is even,  $k \geq 6$ , and let  $m = n/2$ . The construction consists of  $2m + 1$  consecutive blocks,  $A_0, B_1, A_1, B_2, \dots, B_m, A_m$ , in this order, from bottom to top.

For every  $i$ ,  $1 \leq i \leq m$ , let  $u_i$  be a vertex on the lower boundary of  $A_{i-1}$  and let  $v_i$  be a vertex on the lower boundary of  $B_i$ . Let  $\alpha_i$  be an edge joining  $u_i$  and  $v_i$ . The block  $B_i$  is a  $k$ -spiral, and both of its cables are formed by  $\alpha_i$ . For  $2 \leq i \leq m$ , the block  $A_i$  is a 3-spiral, one cable is formed by  $\alpha_i$ , the other one is formed by a folded curve  $\alpha_{i-1}$  (that is, two intervals of  $\alpha_{i-1}$ ). Any curve  $\kappa$  from  $A_{i-2}$  to  $A_i$  has to cross  $\alpha_{i-1}$  at least  $k - 1$  times in  $B_{i-1}$ , and  $\alpha_i$  at least  $k - 1$  times in  $B_i$ . In  $A_{i-1}$ , the curve  $\kappa$  also crosses one of the curves  $\alpha_i$  or  $\alpha_{i-1}$  at least twice, since  $\alpha_{i-1}$  is folded in  $A_{i-1}$ . It follows that when we extend this graph to a saturated  $k$ -complicated topological graph, each vertex has degree at most five.

Note that using  $\lfloor k/2 \rfloor$ -spirals in place of the 3-spirals, we obtain a  $(k, l)$ -saturated  $k$ -complicated topological graph with  $l = 5k/3 - O(1)$ .

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